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ESTIMATION FROM ACCELERATED LIFE TESTS

Richard E. Barlow and Ernest M. Scheuer

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PREFACE

This RAND Memorandum is devoted to the examination and development of new statistical techniques for analyzing accelerated life test data. It is a part of RAND's continuing interest in the theoretical and practical aspects of reliability.

This Memorandum is addressed to statisticians, data analysts, reliability engineers, and others interested in the analysis of accelerated life test data.

Professor Richard E. Barlow, University of California, Berkeley, is a consultant to The RAND Corporation.

SUMMARY

This Memorandum examines and develops new techniques for analyzing life test data. It obtains estimates for the life distribution in the use environment based on data from both the use and the accelerated environments. The techniques require only physically plausible assumptions, not the usual ones involving specification of a family of parametric probability distributions. Procedures are given for testing those assumptions that are made.

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1. INTRODUCTION

It is common practice in certain life testing experiments to subject test items to overstress conditions. This may take many different forms. For example, one might subject electronic capacitors to a high voltage, ball bearings to a high load, or a mechanical assembly to a strong vibration. The purpose is to shorten the time to failure. The problem is to predict the time to failure in the normal or use environment on the basis of such accelerated life test data. [See Winter, Denison, Hietala and Greene (1964) for engineering details and bibliography.]

Let Y (X) be a random variable with distribution G (F) where Y (X) denotes the time to failure under normal (accelerated) conditions. Suppose that F and G are related by a time transformation $\alpha(t)$, where

$$F(t) = G[\alpha(t)]$$

so that, assuming G^{-1} exists,

$$\alpha(t) = G^{-1}F(t) \ .$$

Many authors [e.g., Bessler, Chernoff, and Marshall (1962)] assume

$$\alpha(t) = \alpha \cdot t \ ,$$

i.e., $\alpha(t)$ is a scale transformation of the time axis. They also assume that F and G belong to specified parametric families--usually the exponential distribution. In certain cases there seems to be some justification for assuming that the form of the acceleration function (i.e., the time transformation) is known. Examples are:

1. heaters for vacuum tubes [R. L. Guild (1952)];
2. ball bearings [G. Lieblein and M. Zelen (1956)];
3. paper capacitors [G. J. Levenbach (1957)]; and
4. certain transistors [G. A. Dodson and B. T. Howard (1961)].

The novelty of the problem of course vanishes when $\alpha(t)$ is assumed known and the problem becomes a classical life testing problem. In this study, we do <u>not</u> assume that $\alpha(t)$ is known--only that $\alpha(t)$ and the life distributions satisfy certain geometric restrictions which we believe are intuitively acceptable and reasonable for many applications. However, we do assume that some sample data from items tested in the use environment are available, albeit possibly scanty.

In this paper, we develop least squares estimators for life distributions based on the sampling distribution assuming that the failure rate is increasing on the average. Given life test data from the accelerated and unaccelerated modes, we simultaneously estimate both distributions assuming only that the life distributions have increasing failure rate on the average and that accelerated items tend to fail sooner than unaccelerated items. Applications to fatigue data and accelerated life tests on duplex capacitors are considered. Procedures for testing the underlying assumptions are described.

2. A STATISTICAL MODEL FOR ACCELERATED LIFE TESTING

We assume that observations $\underline{Y} = (Y_1, Y_2, \ldots, Y_m)$ are available on items in the unaccelerated mode having distribution G. These may be results from a truncated or censored life test. It is essential, however, that we have some failure information on life times in the unaccelerated mode. In the accelerated mode it will normally be possible to obtain many more observations, say $\underline{X} = (X_1, X_2, \ldots, X_n)$ from a distribution F also unknown, where again these may result from a censored or truncated test. There are certain natural restrictions which we would want to impose on the time transformation, α, relating F and G. Since acceleration reduces the time to failure, we would expect $\alpha(t) \geq t$. Clearly, α will also be nondecreasing. On the basis of these properties alone we could consider the problem of obtaining an estimate for G which maximizes the joint likelihood and satisfies the restrictions. This problem has been solved by Brunk, Franck, Hanson, and Hogg (1966). However, these estimates impose no restrictions on the failure distributions. Utilization of additional a priori information concerning the life distributions should result in improved estimates.

Birnbaum, Esary, and Marshall (1966) have characterized the smallest class of failure distributions containing the exponential distributions which is closed under the formation of coherent structures. This class is precisely the class of distributions with increasing failure rate average (IFRA); i.e., if the failure rate r(t) exists, then $\frac{1}{t}\int_0^t r(u)du$ is nondecreasing in t. More generally, a distribution F is IFRA if

$\frac{-\log[1 - F(t)]}{t}$ is nondecreasing in $t \geq 0$. Obviously, this includes the class of distributions with nondecreasing failure rate and hence the exponential distribution.

STATEMENT OF THE PROBLEM

Given ordered observations $Y_1 \leq Y_2 \leq \ldots \leq Y_m$ from a distribution G (the unaccelerated life distribution) and ordered observations $X_1 \leq X_2 \leq \ldots \leq X_n$ from a distribution F (the accelerated life distribution), we calculate the empirical distributions G_m and F_n corresponding to G and F, respectively. We wish to obtain estimates $\hat{G}_m$ and $\hat{F}_n$ such that

1. $\hat{G}_m$ and $\hat{F}_n$ are IFRA;
2. $\hat{G}_m(x) \leq \hat{F}_n(x)$ for all x (i.e., stochastic ordering);
3. $\hat{G}_m$ and $\hat{F}_n$ are closest to G_m and F_n, respectively, in a least squares sense, which will be made precise in Sec. 5.

$\hat{G}_m$ will then be our estimate for the life distribution in the unaccelerated mode using data from both the accelerated and unaccelerated environments.

Since in applications one does not always have complete life test data, we also consider estimation procedures for incomplete data.

3. LEAST SQUARES ESTIMATES FOR IFRA DISTRIBUTIONS

Assume that each of n units is observed over some or all of its life. Thus unit i is observed until it either fails at a random age T_i or has attained age L_i (is lost to observation), whichever occurs first ($i = 1, 2, \ldots, n$). The L_i, called limits of observation, are constants or values of other random variables, which are assumed to be independent of T_i. In accelerated life testing censored and/or truncated samples are especially common, so that it is important to consider such data. We now give a procedure for obtaining an estimate of the unknown failure distribution, F, assuming that $-\log \overline{F}(t)/t$ is nondecreasing on $[0, \infty)$ where $\overline{F}(t) = 1 - F(t)$.

3.1 PROCEDURE FOR OBTAINING THE LEAST SQUARES ESTIMATES

(1) Suppose k ($1 \le k \le n$) failures are actually observed. Let $X_1 \le X_2 \le \ldots \le X_k$ denote the ordered ages at failure; for convenience, define $X_o = 0$. Let n_j be the number of items under observation just before X_j, δ_j be the number of failures at X_j, and ℓ_j be the number of "losses" in $[X_{j-1}, X_j)$. Then $n_{j+1} = n_j - \delta_j - \ell_{j+1}$. Ordinarily, δ_j will be one, but may be more than one if the sample contains tied observations. If there are no losses or ties, $n_j = n - j + 1$. Define the product-limit estimate as

$$\overline{F}_n(t) = \begin{cases} 1, & 0 \le t < X_1 \\ \prod_{j=1}^{i} (n_j - \delta_j)/n_j, & X_i \le t < X_{i+1},\ i = 1, \ldots, E-1 \\ 0^{\dagger}, & t \ge X_E \end{cases} \tag{3.1}$$

where $E = \begin{cases} k \text{ if } k < n \\ n \text{ if } k = n \end{cases}$.

† Undefined if $k < n$.

This estimate coincides with the usual empirical sampling distribution estimate if there are no losses and if $k = n$. [See Kaplan and Meier (1958).]

(2) We obtain a least squares estimate, $\hat{F}_n(t)$, for the failure distribution, F, requiring the estimate to have the IFRA property assumed for the distribution. Let

$$\lambda_n(X_i) = \frac{-\log \overline{F}_n(X_i)}{X_i} \qquad \text{for } i = 1, 2, \ldots, E$$

and define

$$\hat{\lambda}_n(X_i) = \max_{s \le i} \min_{t \ge i} \frac{\sum_{j=s}^{t} \lambda_n(X_j) F_n\{X_j\}}{\sum_{j=s}^{t} F_n\{X_j\}} \tag{3.2}$$

where

$$F_n\{X_j\} = \overline{F}_n(X_j -) - \overline{F}_n(X_j +) \ .$$

It is easy to verify from (3.2) that $\hat{\lambda}_n(X_i)$ will be nondecreasing in i. In Sec. 5 we show that this estimate is closest to $\lambda_n(t)$ in a least squares sense. Any nondecreasing function defined at order statistics as in (3.2) is obviously a permissible estimate of λ. However, for definiteness, we define

$$\hat{\lambda}_n(t) = \begin{cases} 0 & , \ t < X_1 \\ \hat{\lambda}_n(X_i), & X_i \le t < X_{i+1}, \quad i = 1, \ldots, E-1 \\ +\infty & , \ t \ge X_E . \end{cases} \tag{3.3}$$

Our estimate for the distribution function will then be

$$\hat{\bar{F}}_n(t) = \begin{cases} 1 & , \quad t < X_1 \\ \exp[-\hat{\lambda}_n(X_i)t] & , \quad X_i \le t < X_{i+1},\ i = 1, \ldots, E-1 \\ 0 & , \quad t \ge X_E. \end{cases} \tag{3.4}$$

3.2 PROCEDURE FOR OBTAINING STOCHASTICALLY ORDERED ESTIMATES

Given observed failures $X_1 \le X_2 \le \ldots \le X_k$ $(k \le n)$, as before, from a sample of size n from a distribution F, and observed failures $Y_1 \le Y_2 \le \ldots \le Y_r$ $(r \le m)$ from a sample of size m from a distribution G, we wish to simultaneously estimate F and G assuming:

i) F and G are IFRA;

ii) $\bar{F}(t) \le \bar{G}(t)$ for all $t \ge 0$.

Let F_n and G_m denote the product limit estimates for F and G as previously described (cf. Eq. (3.1)). Let $\lambda_n(X_i) = -\log \bar{F}_n(X_i)/X_i$ and $\gamma_m(Y_j) = -\log \bar{G}_m(Y_j)/Y_j$ denote estimates of $\lambda(X_i) \equiv \lambda_i$ and of $\gamma(Y_j) \equiv \gamma_j$, respectively. We wish to determine the λ_i and γ_j which, subject to the constraints (3.6), minimize

$$\sum_{i=1}^{k} [\lambda_i - \lambda_n(X_i)]^2\, F_n\{X_i\} + \sum_{j=1}^{r} [\gamma_j - \gamma_m(Y_j)]^2\, G_m\{Y_j\}. \tag{3.5}$$

The minimization must be performed subject to the constraints

$$\begin{aligned} &0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k, \\ &0 \le \gamma_1 \le \gamma_2 \le \ldots \le \gamma_r, \end{aligned} \tag{3.6}$$

and

$$\hat{\bar{G}}_n(t) \ge \hat{\bar{F}}_n(t).$$

To understand the nature of the stochastic ordering restriction it will be convenient to relabel the Y_j's. Let $Y_{11} \leq \ldots \leq Y_{1j_1} < X_1$ denote the first block of Y_j's--those less than X_1. The set may be empty. Similarly let $X_{i-1} \leq Y_{i1} \leq \ldots \leq Y_{ij_i} < X_i$ denote the block of Y_j's in the interval (X_{i-1}, X_i).

Our problem is to minimize (3.5) subject to (3.6) and $Y_{ij_i} \leq \lambda_i$ $(i = 1, 2, \ldots, k)$. This is a straightforward quadratic programming problem. A RAND code (RSQPF4) is available for solving this problem. In some applications we can obtain the solution to our problem in a manner similar to that in which we obtained the IFRA estimate previously. To be specific, consider all possible "interlacings" of the Y_j's among the λ_i's. For each such linear ordering we can obtain an explicit solution to the problem of minimizing (3.5), subject to the linear ordering, using Theorem 1 of Sec. 5. Now choose that solution corresponding to a linear ordered constraint set which has minimum sum of squares. (Unfortunately, for even moderate k and r, the number of interlacings that must be considered will be quite large.)

3.3 APPLICATION TO FATIGUE DATA

To illustrate the ideas discussed above we consider some fatigue data extracted from a paper by R. P. Felgar (1963). Specimens of 7075-T6 aluminum alloy were subjected to cyclic loading under normal air pressure and under near-vacuum conditions. The data are given in Table 1.

In this application, near-vacuum (simulating outer space) constitutes the use environment and normal air pressure the accelerated environment. The distribution of cycles-to-failure in vacuum is

Table 1

ALLOY FATIGUE DATA

Accelerated Environment (Air)		Use Environment (Vacuum)	
Specimen	10^5 Cycles-to-Failure	Specimen	10^5 Cycles-to-Failure
X_1	1.23	Y_1	3.3
X_2	1.41	Y_2	3.5
X_3	1.62	Y_3	14.9
X_4	2.08	Y_4	40.7
X_5	2.09	Y_5	45.7
X_6	2.30	Y_6	46.8
X_7	2.82	Y_7	58.2

denoted by G, and in normal atmosphere by F.

In our example the number of observations in the use environment and in the accelerated environment happen to be the same. Often there will be fewer observations available in the use environment.

Using formula (3.2) to estimate the $\lambda_n(X_i)$'s and, <u>mutatis mutandis</u>, the same formula to estimate the $\gamma_m(Y_i)$'s we obtain

$$\hat{\lambda}_7(X_1) = .1253$$
$$\hat{\lambda}_7(X_2) = .2386$$
$$\hat{\lambda}_7(X_3) = .3075$$
$$\hat{\lambda}_7(X_4) = .4074$$
$$\hat{\lambda}_7(X_5) = .5994$$
$$\hat{\lambda}_7(X_6) = .8460$$

and

$$\hat{\gamma}_7(Y_i) = .0450, \quad i = 1, \ldots, 6.$$

The estimates $\hat{\hat{F}}_7(t)$ and $\hat{\hat{G}}_7(t)$ are graphed in Figs. 1 and 2. Since these estimates are already stochastically ordered, the accelerated observations give us no further information concerning the distribution G. A statistical test for IFRA, discussed in Sec. 4, does not reject the hypothesis that G is an exponential distribution. This is also made credible by our estimate $\hat{\hat{G}}_7$. The accelerated data on the other hand are significantly IFRA according to the total-time-on-test statistic discussed in Sec. 4.

3.4 ACCELERATED LIFE TESTS ON DUPLEX CAPACITORS

The data in Table 2 were obtained from conducting accelerated life tests on duplex capacitors. Acceleration is due to a voltage stress at room temperature. The criterion for failure is the leakage current's exceeding a certain level. The times to failure are not exact, but approximately close to the actual failure times. To illustrate our procedures, we consider data at only three voltage levels.

Note that the 50-volt test was censored at the ninth failure. Using the total-time-on-test statistic described in Sec. 4, the 50-volt and 70-volt data were not significantly IFRA relative to the hypothesis of exponentiality. They appear, in fact, to follow an exponential distribution. The 60-volt data, on the other hand, are significantly IFRA even at the .001 significance level.

As we would expect, the data seem to be stochastically decreasing with respect to increasing voltage. A statistical test for stochastic ordering, given censored data and using a modified Wilcoxon statistic, is discussed in the next section.

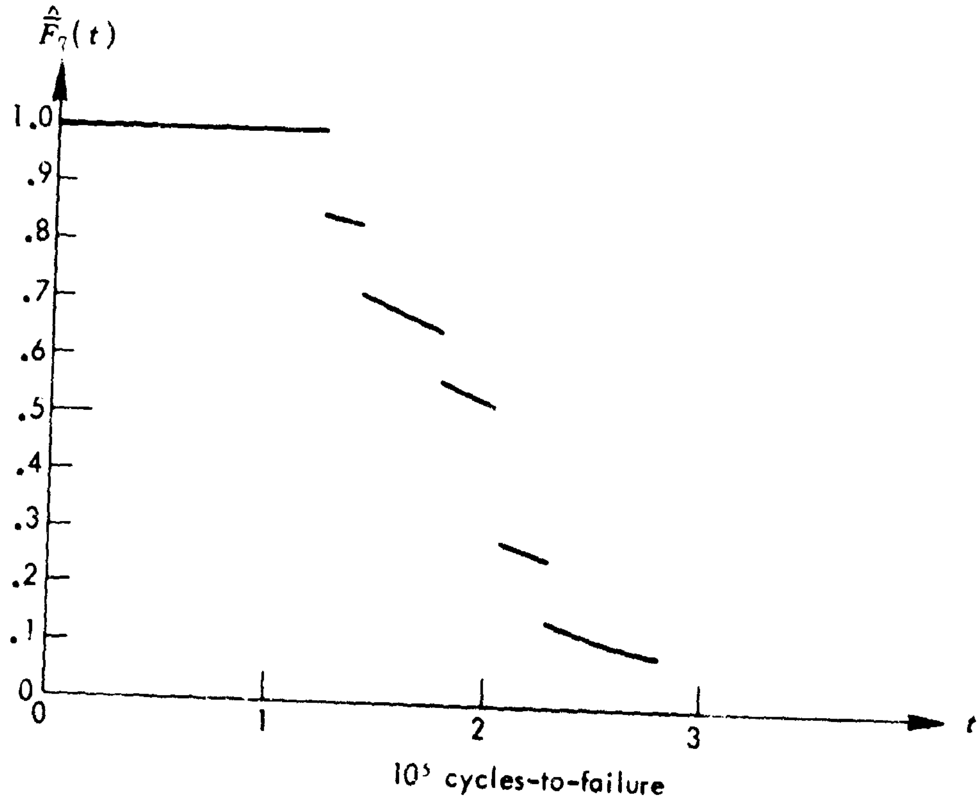


Fig. 1 -- Data from accelerated environment

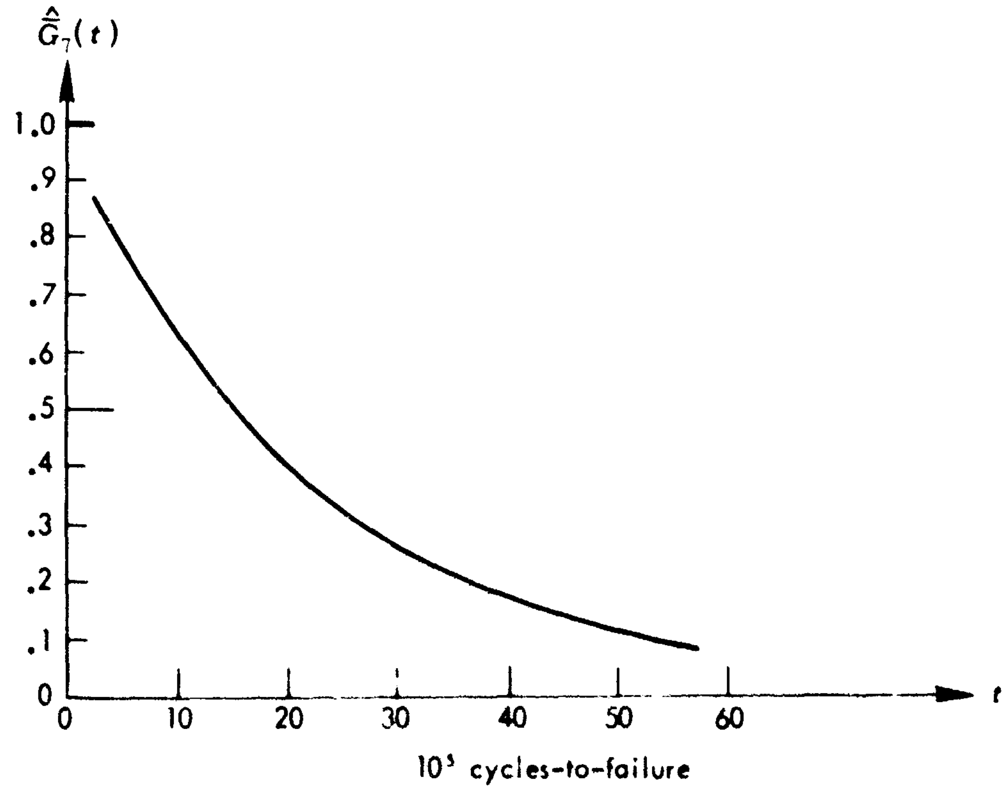


Fig. 2 -- Data from use environment

Table 2

DUPLEX CAPACITOR ACCELERATED LIFE DATA

50 Volts (Room Temp.)	60 Volts (Room Temp.)	70 Volts (Room Temp.)
n = 13	n = 13	n = 14
Times to Failure (hrs.)	Times to Failure (hrs.)	Times to Failure (hrs.)
526	56	3
1075	517	526
1075	517	526
1794	517	932
2080	517	1023
2891	517	1193
2891	517	1223
4211	758	2085
4211	758	2296
	758	2442
	758	4216
	1016	4216
	1186	4216
		5589

The IFRA estimates of the survival probability function, using Eq. (3.4), for the 50-, 60-, and 70-volt data are graphed in Figs. 3, 4, and 5, respectively.

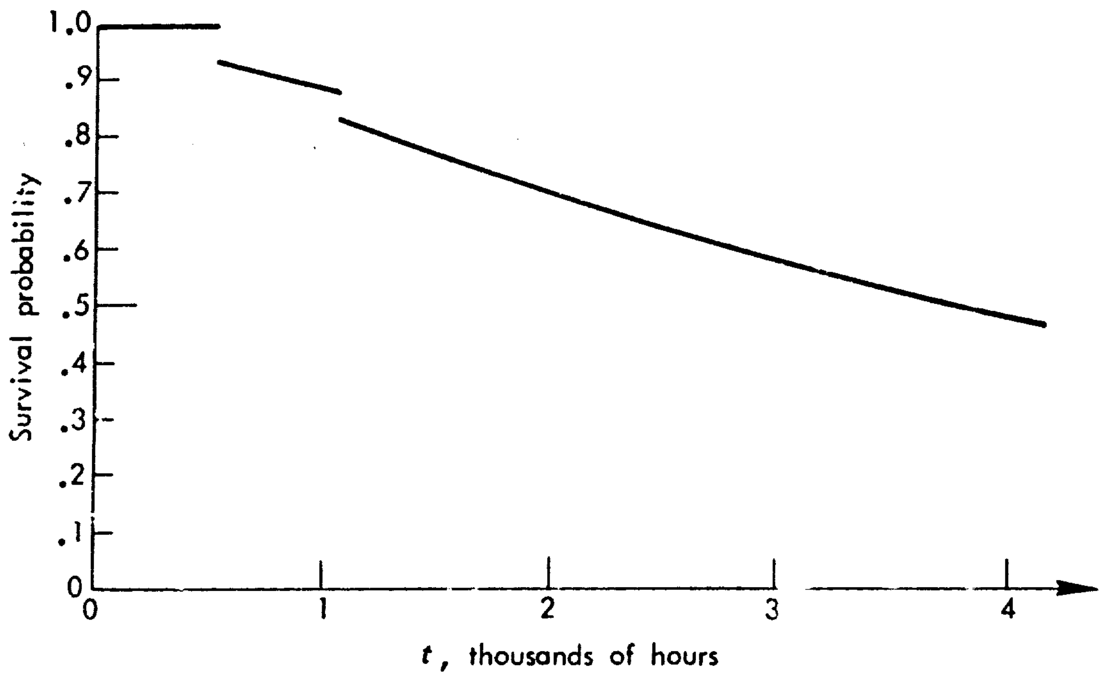


Fig. 3 -- 50-volt data

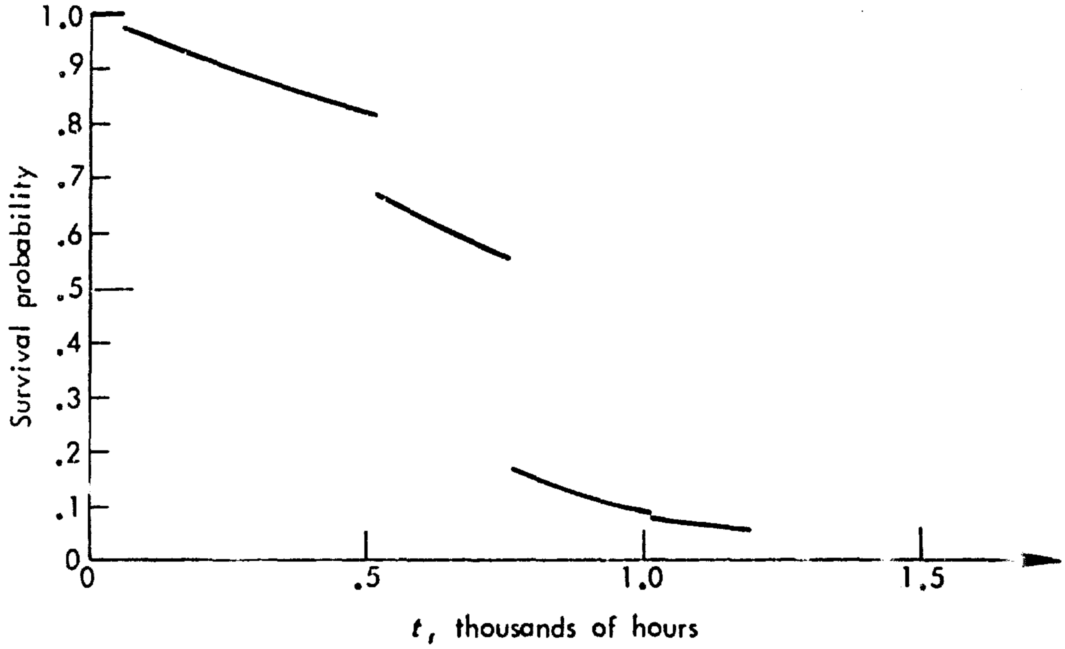


Fig. 4 -- 60-volt data

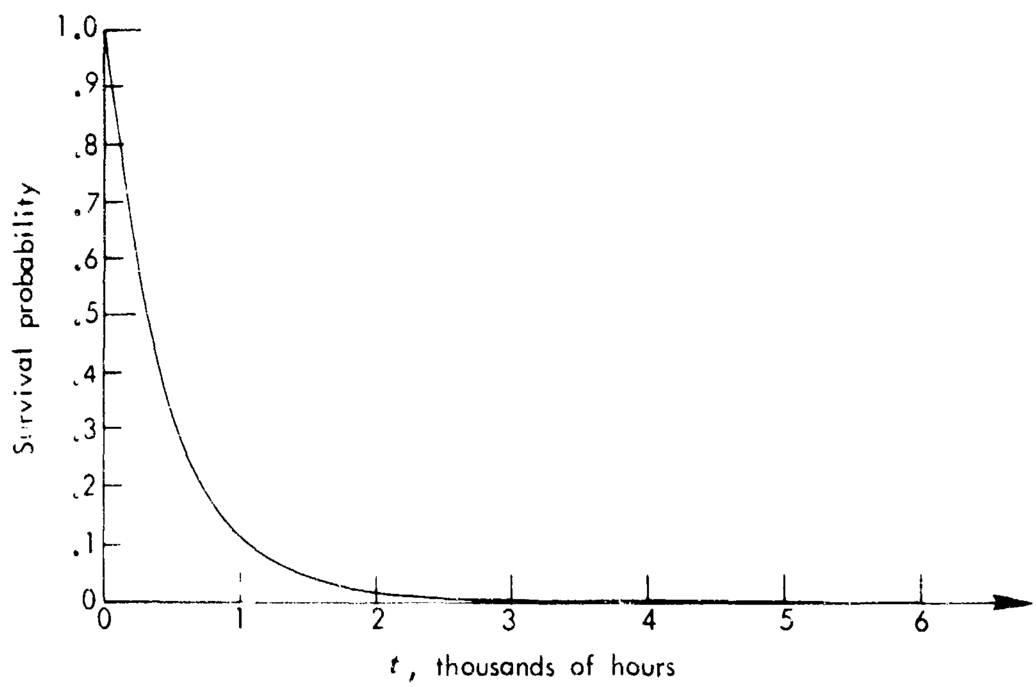


Fig. 5 -- 70-volt data

4. STATISTICAL TESTS FOR IFRA (DFRA)† AND STOCHASTIC ORDERING

Before applying the techniques of this paper to life test data, it is first necessary to check the validity of the assumptions made. A test for IFRA (or DFRA) has been considered by Barlow (1968) and is described below. (The asymptotic relative efficiency of this test against various alternatives has been investigated by Bickel and Doksum (1967).) Suppose the first r failures out of a sample of size n are recorded. Let $X_1 \leq X_2 \leq \ldots \leq X_r$ denote the observed failure times. Note that "withdrawals" may occur between X_i and X_{i+1} and that r is, in general, a random variable. Let n(u) be the (random) number of items on test at time u. The total time on test up to the i^{th} order statistic is

$$T(X_i) = \int_0^{X_i} n(u)du.$$

The total-time-on-test statistic is $\sum_{i=1}^{r-1} T(X_i)/T(X_r)$. Under the exponential hypothesis

$$Z = \frac{\sum_1^{r-1} T(X_i) - \frac{(r-1)}{2} T(X_r)}{T(X_r)\sqrt{(r-1)/12}}$$

is approximately N(0,1) even for relatively small r. If the distribution of time to failure is IFRA (DFRA) and $Y_1 \leq Y_2 \leq \ldots \leq Y_r$ denotes an independent ordered sample from an exponential distribution, then

$$\frac{\sum_1^{r-1} T(X_i)}{T(X_r)} \underset{st}{\geq} \left(\underset{st}{\leq}\right) \frac{\sum_{i=1}^{r-1} T(Y_i)}{T(Y_r)}$$

† Decreasing failure rate average. Cf. Sec. 2.

where $\geq_{st}$ denotes stochastic ordering. Hence, a natural test rejects exponentiality in favor of IFRA if

$$\frac{\sum_{1}^{r-1} T(X_i)}{T(X_r)} \geq c_\alpha$$

where c_α is defined by

$$P_G\left\{\frac{\sum_{1}^{r-1} T(Y_i)}{T(Y_r)} \geq c_\alpha\right\} = \alpha$$

and $G(x) = 1 - e^{-x}$ for $x \geq 0$.

Table 3 presents Z values corresponding to the failure data in Tables 1 and 2.

Table 3

Z VALUES CALCULATED FROM THE DATA OF TABLES 1 AND 2

Data	Z Values
Fatigue Data at air pressure (Table 1)	2.8397
Fatigue Data at vacuum conditions (Table 1)	0.4984
Duplex capacitors at 50 volts (Table 2)	0.7837
Duplex capacitors at 60 volts (Table 2)	3.2152
Duplex capacitors at 70 volts (Table 2)	0.5511

The conclusions concerning the tests of exponentiality versus IFRA for the alloy data (Sec. 3.3) and for the capacitor data (Sec. 3.4) were reached by comparing the Z values in Table 3 with percentage points of the standard normal distribution.

TESTS FOR STOCHASTIC ORDERING

Although stochastic ordering may be obvious from some accelerated life test data, there may be instances when a statistical test is appropriate. For example, it is not clear that there is the desired stochastic ordering between the 60-volt duplex capacitor data and the 70-volt capacitor data.

A generalized Wilcoxon test for comparing arbitrarily singly-censored samples has been proposed by E. A. Gehan (1965 a,b). We assume that m and n items are subject to life test and we observe

$$\begin{array}{ll} X_1', \ldots, X_{r_1}' & r_1 \text{ censored} \\ X_{r_1+1}, \ldots, X_n & n-r_1 \text{ failures} \\ Y_1', \ldots, Y_{r_2}' & r_2 \text{ censored} \\ Y_{r_2+1}, \ldots, Y_m & m-r_2 \text{ failures.} \end{array}$$

(These observations are not necessarily ordered.)

Define

$$U_{ij} = \begin{cases} -1 & \text{if } X_i < Y_j \text{ or } X_i \le Y_j' \\ 0 & \text{if } X_i = Y_j \text{ or } X_i' < Y_j \text{ or } Y_j' < X_i \text{ or } (X_i', Y_j'), \text{ i.e., both are censored} \\ +1 & \text{if } X_i > Y_j \text{ or } X_i' \ge Y_j \end{cases}$$

and calculate the statistic

$$W = \sum_{i,j} U_{ij}$$

where the sum is over all nm comparisons. To test the hypothesis

$$H_o : X \underset{st}{=} Y$$

versus

$$H_1 : X \underset{st}{\leq} Y$$

we reject H_o if

$$Z = \frac{W}{\sqrt{\{\mathrm{Var}\ (W \mid P, H_o)\}}} < c_\alpha$$

where c_α is determined using the fact that Z is asymptotically N(0,1).

$$\mathrm{Var}\ (W \mid P, H_o) = \frac{nm}{(n+m)(n+m-1)} \left\{ \sum_{i=1}^{s} m_i M_i (M_{i-1}) + \sum_{i=1}^{s} \ell_i M_i (M_i + 1) \right.$$

$$\left. + \sum_{i=1}^{s} m_i (m+n-M_i-L_{i-1})(n+m-3M_{i-1}-m_i-L_{i-1}-1) \right\}$$

where

$$M_j = \sum_{i=1}^{j} m_i, \text{ with } M_o = 0$$

$$L_j = \sum_{i=1}^{j} \ell_i, \text{ with } L_o = 0$$

and

m_i = number of uncensored observations at rank i in rank ordering of uncensored observations with distinct values (i = 1,2, ..., s);

ℓ_i = number of right-censored observations with values greater than observations at rank i but less than observations at rank (i+1).

Var (W | P,H_o) is the variance of W under H_o and conditioned on the pattern of observations, P.

5. OPTIMALITY OF LEAST SQUARE IFRA ESTIMATORS

In Sec. 3 we described a method for estimating an IFRA distribution. Since Marshall and Proschan (n.d.) have shown that the maximum likelihood estimate of F assuming F is IFRA is <u>not</u> consistent, we have employed a least squares criterion which <u>is</u> consistent. Under the IFRA assumption $\lambda(t) = \frac{-\log \overline{F}(t)}{t}$ is nondecreasing. Given the observed failure times $X_1 \leq X_2 \leq \ldots \leq X_k$ based on a random sample from F we seek a "good" estimate of $\lambda(t)$. Letting F_n denote the product-limit estimate (Eq. (3.1)), then

$$\lambda_n(X_i) = \frac{-\log \overline{F}_n(X_i)}{X_i}$$

provides an initial estimate of $\lambda(X_i)$ which will enjoy all of the properties of the product-limit estimate; e.g., it will be strongly consistent. It is also known that F_n is the maximum likelihood estimate of F in the class of all distributions (see Kaplan and Meier (1958)) and hence λ_n is the unrestricted maximum likelihood estimate for λ.

Define

$$\hat{\lambda}_n(X_i) = \underset{s\leq i}{\text{Max}}\ \underset{t\geq i}{\text{Min}} \frac{\sum_{j=s}^{t} \lambda_n(X_j)F_n\{X_j\}}{\sum_{j=s}^{t} F_n\{X_j\}} .$$

Clearly, $\hat{\lambda}_n$ will be nondecreasing at order statistics. Furthermore, if λ is any other increasing function, then $\hat{\lambda}_n$ has the property that

$$\sum_{i=1}^{n} [\lambda_n(X_i) - \lambda(X_i)]^2 F_n\{X_i\} = \sum_{i=1}^{n} [\lambda_n(X_i) - \hat{\lambda}_n(X_i)]^2 F_n\{X_i\} + \sum_{i=1}^{n} [\hat{\lambda}_n(X_i) - \lambda(X_i)]^2 F_n\{X_i\}. \tag{5.1}$$

Hence

$$\sum_{i=1}^{n} [\hat{\lambda}_n(X_i) - \lambda_n(X_i)]^2 F_n\{X_i\} \le \sum_{i=1}^{n} [\lambda(X_i) - \lambda_n(X_i)]^2 F_n\{X_i\}$$

and therefore $\hat{\lambda}_n$ is closest to λ_n in the least squares sense with respect to F_n in the class of nondecreasing functions. A proof of inequality (5.1) can be found in Brunk (1965) and Marshall and Proschan (1965, Theorem 5.1). For completeness we present a proof for the discrete case.

Theorem 1: (Marshall and Proschan)

Let h_i $(i = 0, 1, \ldots, n)$ be a nondecreasing sequence, g_i an arbitrary sequence, $m_i \ge 0$, $i = 0, 1, 2, \ldots, n$ and

$$\tilde{g}_i = \operatorname*{Max}_{s \le i} \operatorname*{Min}_{t \ge i} \frac{\sum_{j=s}^{t} g_j m_j}{\sum_{j=s}^{t} m_j} ;$$

then

$$\sum_{i=0}^{n} (g_i - h_i)^2 m_i \ge \sum_{i=0}^{n} (\tilde{g}_i - h_i)^2 m_i + \sum_{i=0}^{n} (g_i - \tilde{g}_i)^2 m_i ,$$

i.e., in the class of increasing functions, $\tilde{g} = (\tilde{g}_0, \ldots, \tilde{g}_n)$ is closest to $g = (g_0, \ldots, g_n)$ in the least squares with respect to the measure $\{m_i\}_{i=0}^{n}$.

Proof:

Clearly, $\tilde{g}_i$ is increasing in i. We need only show

$$\sum_{i=0}^{n} (\tilde{g}_i - h_i)(g_i - \tilde{g}_i)\, m_i \ge 0.$$

Suppose $\tilde{g}_i$ is constant on the interval $[a, b]$. Then

$$\sum_{i=a}^{b} (\tilde{g}_i - h_i)(g_i - \tilde{g}_i)\, m_i$$

$$= \sum_{i=a}^{b-1} \left\{ \sum_{j=a}^{i} (g_j - \tilde{g}_j) m_j \right\} (h_{j+1} - h_j) - \left\{ \sum_{j=a}^{b} (g_j - \tilde{g}_j) m_j \right\} h_b$$

$$= \sum_{i=a}^{b-1} \left\{ \frac{\sum_{j=a}^{i} g_j m_j}{\sum_{j=a}^{i} m_j} - \tilde{g}_j \right\} \left(\sum_{j=a}^{i} m_j \right) (h_{j+1} - h_j) + \left\{ \frac{\sum_{j=a}^{b} g_j m_j}{\sum_{a}^{b} m_j} - \tilde{g}_j \right\} \sum_{a}^{b} m_j \, h_b \, .$$

Now

$$\frac{\sum_{j=a}^{i} g_j m_j}{\sum_{j=a}^{i} m_j} \geq \operatorname*{Min}_{t \geq i} \frac{\sum_{j=a}^{t} g_j m_j}{\sum_{j=a}^{t} m_j} = \operatorname*{Max}_{s \leq i} \operatorname*{Min}_{t \geq i} \frac{\sum_{j=s}^{t} g_j m_j}{\sum_{j=s}^{t} m_j} = \tilde{g}_i \, . ||$$

Any nondecreasing function defined at order statistics as in (3.2) is obviously permissible. However, for definiteness, we define

$$\hat{\lambda}_n(t) = \begin{cases} 0, & t < X_1 \\ \hat{\lambda}_n(X_i), & X_i \leq t < X_{i+1} \\ +\infty, & t \geq X_n \, . \end{cases}$$

Note that this estimate will be closest to $\lambda_n(t)$ for all $0 \leq t < X_n$ when $\hat{\lambda}_n(X_i) = \lambda_n(X_i)$. Our estimate for the distribution function will then be

$$\hat{\bar{F}}_n(t) = \begin{cases} 1, & t < X_1 \\ \exp[-\hat{\lambda}_n(X_i)t], & X_i \le t < X_{i+1} \\ 0, & t \ge X_n. \end{cases}$$

The strong consistency of $\hat{F}_n$ is inherited from the strong consistency of F_n. By Corollary 3.3 of Brunk (1965) it follows that $|\lambda_n(X_i) - \lambda(X_i)| \le \varepsilon$ for $i = 1,2, \ldots, n$ implies $|\hat{\lambda}_n(X_i) - \lambda(X_i)| < \varepsilon$ for $i = 1,2, \ldots, n$. Strong consistency follows from the strong consistency of λ_n.

The additional inequalities required by the stochastic ordering assumption, $G(x) \le F(x)$, are immediately evident upon examining the graphs of $-\log \hat{\bar{G}}_n(t)$ and $-\log \hat{\bar{F}}_n(t)$.

6. IFRA TIME TRANSFORMATIONS

The stochastic ordering assumption on the time transformation $\alpha(t) \geq t$ is probably the weakest and most intuitive requirement on $\alpha(t)$. By making a stronger (but more difficult to justify) assumption on $\alpha(t)$ we should obtain "improved" estimates.

We adopt the following definition of an IFRA time transformation:†

Definition

$\alpha(t)$ is an IFRA time transformation if $\alpha(t)/t$ is nondecreasing in $t \geq 0$ and $\alpha(t) \geq t$.

This definition would be perhaps quite readily acceptable if $\alpha(t)$ depended only on the environment and not on the original distribution G. Of course in practice this seems somewhat unlikely. However, this definition leads to a class of time transformations that seems intuitively reasonable.

Lemma 1.

$\alpha(t)$ is an IFRA time transformation if and only if

(i) $\alpha(t)$ is nondecreasing in $t \geq 0$;

(ii) $\alpha(t) \geq t$

(iii) for every IFRA G, we have that $F(t) = G[\alpha(t)]$ is IFRA.

Proof:

(a) Suppose $\alpha(t)$ is an IFRA time transformation. Then $-\log \overline{F}(t)/t = [-\log \overline{G}(\alpha(t))/\alpha(t)][\alpha(t)/t]$ is nondecreasing in $t \geq 0$ since $\alpha(t)$ is nondecreasing in $t \geq 0$. I.e., F is IFRA.

---

†We could alternatively consider IFR time transformations. A distribution F is IFR if $\log[1 - F(t)]$ is concave on its support.

(b) Suppose conditions (i)-(iii) hold. Choose G to be the unit exponential distribution and define $F(t) = G[\alpha(t)]$. Then $\alpha(t)/t =$ $-\log \overline{F}(t)/t$ is nondecreasing in $t \geq 0$ since F is IFRA. ||

Clearly, if $\alpha_1(t)$ and $\alpha_2(t)$ are IFRA time transformations then so is their composition $\alpha_1[\alpha_2(t)]$.

__Lemma 2__

If $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ $(X_1 \leq X_2 \leq \ldots \leq X_n)$ are order statistics from G (F) and $F(t) = G[\alpha(t)]$, where α is an IFRA time transformation, then

$$Y_j - Y_i \overset{st}{\geq} X_j - X_i \qquad \text{for } i \leq j$$

where $\overset{st}{\geq}$ means stochastically greater than. In particular, the sample range from G is stochastically greater than the sample range from F.

__Proof:__

Let $Y_i^* = G^{-1}F(X_i)$ and note $(Y_1^*, \ldots, Y_n^*) \overset{st}{=} (Y_1, \ldots, Y_n)$. Since $\frac{G^{-1}F(t)}{t} = \frac{\alpha(t)}{t} \geq 1$ and is nondecreasing in $t \geq 0$,

$$\frac{Y_j^* - Y_i^*}{X_j - X_i} = \frac{G^{-1}F(X_j) - G^{-1}F(X_i)}{X_j - X_i} \geq 1,$$

which implies the result. ||

From the results of Barlow and Proschan (1966), it is easy to verify that IFRA time transformations stochastically decrease order statistics and the sample coefficient of variation. It follows that population coefficient of variation and moments are also decreased.

Assuming that $\alpha(t)/t = G^{-1}F(t)/t$ is nondecreasing in $t \geq 0$ is equivalent to assuming that $F(t)$ crosses $G(\theta t)$ at most once and from below if at all for every $\theta > 0$. In this sense, F is "sharper" than G.

In analogy with the least squares estimates, we confine attention to estimates of the form G_γ and F_γ where

$$-\log \bar{G}_\gamma(t) = \begin{cases} 0 & , \quad 0 < t < Y_1 \\ t\gamma(Y_i) , & Y_i \leq t < Y_{i+1} \\ \infty & , \quad t > Y_m \end{cases}$$

and $-\log \bar{F}_\lambda(t)$ is defined similarly.

Given observations $Y_1 \leq Y_2 \leq \ldots \leq Y_m$ $(X_1 \leq X_2 \leq \ldots \leq X_n)$ from a distribution G (F) we wish to estimate both F and G assuming

(i) F and G are IFRA,

(ii) $G^{-1}F(t)/t$ is nondecreasing in $t \geq 0$

(iii) $G^{-1}F(t) \geq t$ for $t \geq 0$.

Define $\gamma(t) = -\log \bar{G}(t)/t$ and $\gamma_m(t) = -\log \bar{G}_m(t)/t$ where G_m is the usual empirical distribution or product limit based upon observations $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ from the normal or "use" environment. We consider the problem of minimizing

$$\int_0^\infty [\gamma(x) - \gamma_m(x)]^2 \, dG_m(x) + \int_0^\infty [\lambda(x) - \lambda_n(x)]^2 \, dF_n(x) \qquad (6.1)$$

with respect to γ and λ subject to

$$0 \le \gamma(Y_1) \le \gamma(Y_2) \le \dots \le \gamma(Y_m),$$

$$0 \le \lambda(X_1) \le \lambda(X_2) \le \dots \le \lambda(X_n),$$

$G_\gamma^{-1}F_\lambda(t)/t$ is nondecreasing in $t \ge 0$, and $G_\gamma^{-1}F_\lambda(t) \ge t$. If $\hat{G}_n^{-1}\hat{F}_n(t)$ is nondecreasing in $t > 0$ and $\hat{G}_n^{-1}\hat{F}_n(t) \ge t$ then clearly the least squares solutions $\hat{\gamma}_m$ and $\hat{\lambda}_n$ also minimize (6.1).

To understand the restrictions on the time transformation, we make use of the so-called Q - Q plot (Q for quantile) which is widely used by data analysts [cf. Wilk and Gnanadesikan (1968)]. Let G_γ (F_λ) denote estimates of G (F). A Q - Q plot based on G_γ and F_λ is merely a plot of $G_\gamma^{-1}F_\lambda$. If G_γ and F_λ are the empirical distribution functions, this provides a quick, although heuristic, check on our assumptions concerning $G^{-1}F$. Let $H(x) = 1-e^{-x}$ and consider the graph of $H^{-1}G_\gamma$ and $H^{-1}F_\lambda$ as illustrated in Figs. 6 and 7. Note that $[H^{-1}G_\gamma]^{-1}H^{-1}F_\lambda = G_\gamma^{-1}F_\lambda$.

The parameters associated with line segments are their slopes. The line segments, if extended, would pass through the origin. Fix an ordinate value, say v. Define x(v) to be the largest value of x such that $H^{-1}F_\lambda(x) \le v$ and y(v) to be the largest value of y such that $H^{-1}G_\gamma(y) \le v$. Letting v range over all positive values, plot the

locus of points $(x(v),y(v))$ in Fig. 8. Since $G^{-1}F(t)/t$ is nondecreasing in $t \geq 0$, we demand the same of $G_\gamma^{-1}F_\lambda(t)/t$. Hence $G_\gamma^{-1}F_\lambda$ will consist of line segments with increasing slopes, each greater than or equal to one. Since $G_\gamma^{-1}F_\lambda$ cannot be flat over x-axis intervals, we must have

$$\lambda_1 X_1 \geq \gamma_1 Y_1 \tag{6.2}$$

and

$$\gamma_{m-1} Y_m \geq \lambda_{n-1} X_n \; . \tag{6.3}$$

If we were to plot points $\gamma_{i-1}Y_i$, $\gamma_i Y_i$, $(i = 1,2, \ldots, m)$ and $\lambda_{j-1}X_j$, $\lambda_j X_j$, $(j = 1, 2, \ldots, n)$ on the same line, we would see that intervals of the form $(\gamma_{i-1}Y_i, \gamma_i Y_i)$ must fall within intervals of the form $(\lambda_{j-1}X_j, \lambda_j X_j)$ in order that $G_\gamma^{-1}F_\lambda$ not be constant over intervals.

To minimize (6.1) subject to G, F IFRA and $G^{-1}F(t)/t$ nondecreasing in $t \geq 0$ we must minimize (6.1) subject to the constraints

(i) $\quad 0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m; \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$,

and

(ii) $$\left.\begin{array}{l} \gamma_{i-1}Y_i \geq \lambda_{j_i - 1}X_{j_i} \\ \\ \gamma_i Y_i \leq \lambda_{j_i} X_{j_i} \end{array}\right\} \begin{array}{l} i = 1,2, \ldots, m \\ j = 1,2, \ldots, n \end{array} \qquad (m \leq n)$$

where we must consider every possible positioning sequence $(j_1, j_2, \ldots, j_m)$ in turn. Given a positioning sequence $(j_1, j_2, \ldots, j_m)$ we must also

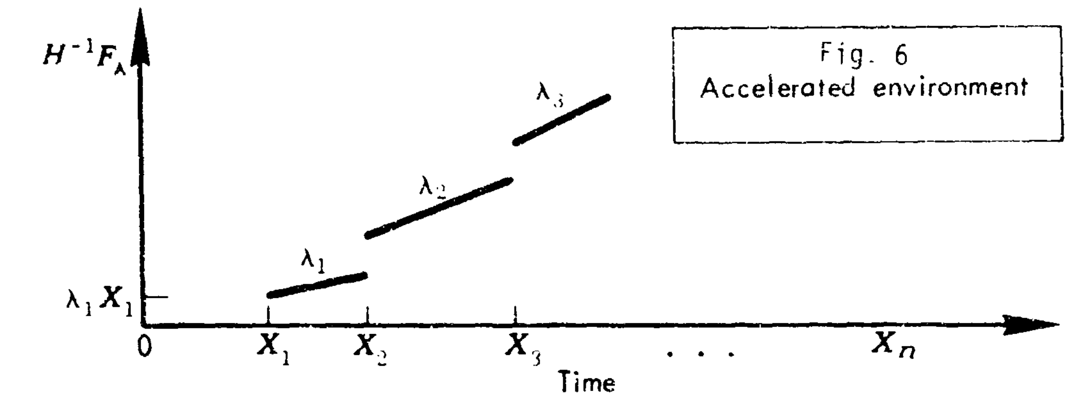


Fig. 6
Accelerated environment

Fig. 7
Use environment

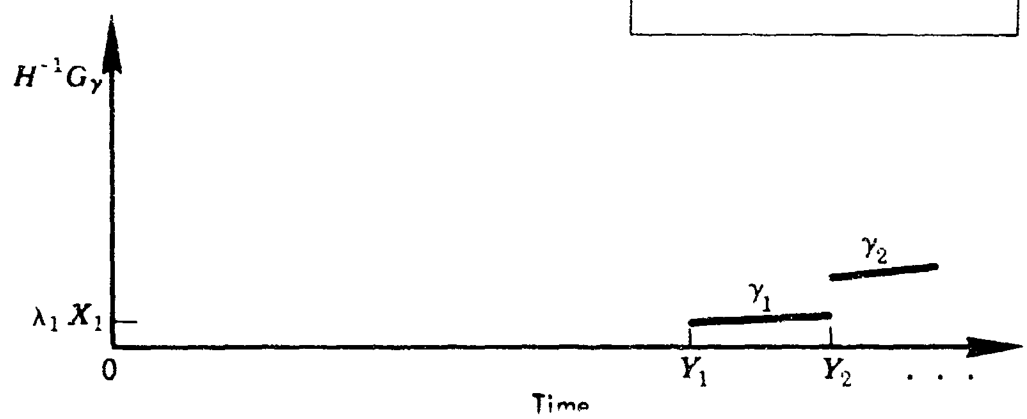
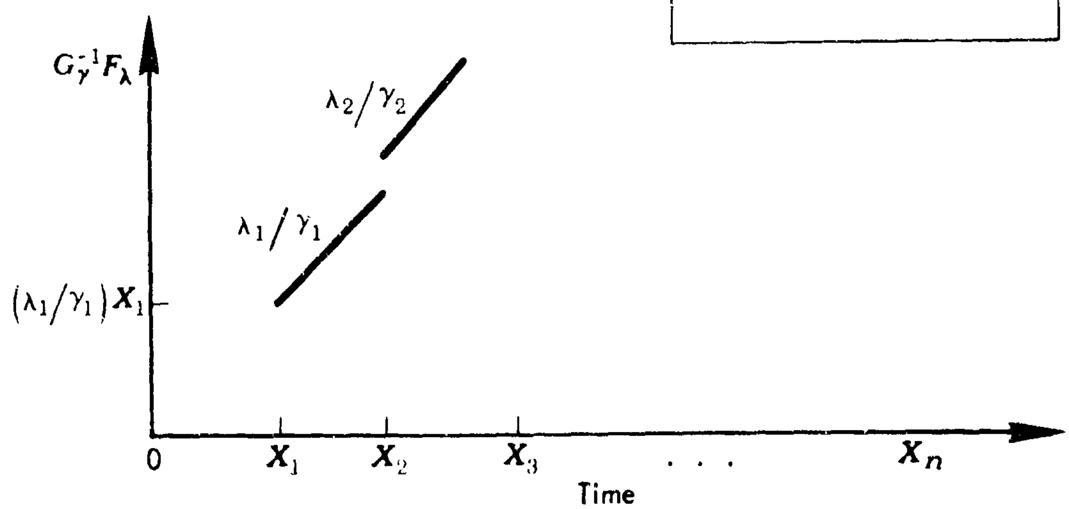


Fig. 8
Q-Q plot



satisfy

(iii) $$\lambda_{j_i} \Big/ \gamma_i \le \lambda_{j_{i+1}} \Big/ \gamma_{i+1} \qquad i = 1,2, \ldots, n-1.$$

A computer program is necessary to solve this problem in general. If n is large so that we have a good estimate of F, then we may let $\lambda_j = \hat{\lambda}_j$ and minimize (6.1) with respect to γ subject to the restrictions on γ.

It is intuitively clear from Fig. 8 that if the $\hat{\gamma}_i$ estimates are too high the slopes $\hat{\lambda}_i/\hat{\gamma}_i$ may not satisfy our requirement $\hat{\lambda}_i/\hat{\gamma}_i \le 1$. Hence the λ_i's will tend to force our estimates of the γ's down.

7. A MODIFIED PROBLEM

By introducing additional restrictions on G, the failure distribution for the use environment, we may obtain an explicit solution to the problem of Sec. 6. Assume

$$\bar{G}(t) = \begin{cases} 1 & , \ t < Y_1 \\ b \exp[-\gamma(t - Y_1)), & t \geq Y_1 \end{cases}$$

where b and γ are parameters to be determined subject to the restriction $e^{-\gamma Y_1} \leq b \leq 1$. The restriction on b insures that G is IFRA. This is a more stringent assumption on G than made previously. However, one can imagine situations where it might be approximately true. Figure 7 is now a single ray starting above Y_1 with slope γ and continuing indefinitely to the right.

We consider the problem of minimizing

$$\int [\gamma - \gamma_m(x)]^2 \, dG_m(x) + \int_0^\infty [\lambda(x) - \lambda_n(x)]^2 \, dF_n(x) \qquad (7.1)$$

with respect to γ and λ subject to

$$0 \leq \gamma \leq \frac{\lambda_1 X_1}{Y_1} \quad \text{and} \quad 1 \leq \frac{\lambda_1}{\gamma} \leq \frac{\lambda_2}{\gamma} \leq \ldots \leq \frac{\lambda_{n-1}}{\gamma} \ .$$

Note that in most practical situations $Y_1 > X_1$ so that $\gamma \leq \frac{\lambda_1 X_1}{Y_1}$ will automatically imply $1 < \frac{\lambda_1}{\gamma}$. Let $\theta_o = Y_1\gamma$ and $\theta_j = \lambda_j X_1$, $j = 1, 2, \ldots, n - 1$.

We now assume that we are dealing with a complete, noncensored sample with no tied observations and no losses. Rewriting (7.1) we

seek to minimize

$$\frac{1}{mY_1^2} \sum_{i=1}^{m} [\theta_o - Y_1\gamma_m(Y_i)]^2 + \sum_{j=1}^{n-1} [\theta_j - X_1\lambda_n(X_j)]^2 \frac{1}{nX_1^2}$$

subject to $\theta_o \leq \theta_1 \leq \ldots \leq \theta_{n-1}$.

Equivalently, we wish to minimize

$$\frac{1}{Y_1^2} \left[\theta_o - Y_1 \frac{\sum_1^m \gamma_m(Y_i)}{2m}\right]^2 + \sum_{j=1}^{n-1} [\theta_j - X_1\lambda_n(X_j)]^2 \frac{1}{nX_1^2}$$

subject to $\theta_o \leq \theta_1 \leq \ldots \leq \theta_{n-1}$. The solution is immediate from Theorem 1 if we make the following identification:

$$g_o = Y_1 \frac{\sum_1^m \gamma_m(Y_i)}{2m} \; ; \; g_j = X_1\lambda_m(X_j) \quad \text{for } j = 1, 2, \ldots, n-1$$

and

$$m_o = \frac{1}{Y_1^2}, \quad m_j = \frac{1}{nX_1^2} \quad \text{for } j = 1, 2, \ldots, n.$$

Applying the method of this Section to the fatigue data of Table 1 we find

$$\hat{\gamma} = .0193$$
$$\hat{\lambda}_1 = .1253$$
$$\hat{\lambda}_2 = .2386$$
$$\hat{\lambda}_3 = .3075$$
$$\hat{\lambda}_4 = .4074$$
$$\hat{\lambda}_5 = .5994$$
$$\hat{\lambda}_6 = .8460$$

Comparing these estimates with those in Sec. 3.3, we note that our estimate of the failure rate in vacuum (the use environment in this case) is less, i.e., $\hat{\gamma}$ = .0193 versus .0450 in Sec. 3.3.

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10. ABSTRACT

A statistical technique for analyzing life test data from tests under overstress conditions. Usually the analyst simply assumes that the accelerated and normal use failure distributions belong to specified parametric families, such as the exponential distribution, and that the test experience is a scale transformation of reality. The new technique assumes only that (1) the failure rate is increasing on the average, and (2) test items in the overstress environment tend to fail sooner than those in normal use. Some data from the use environment are required, but may be scanty. Least squares estimators for the life distributions are developed, using both sets of data; quadratic programming may be used for computation. Procedures are given to test the validity of the assumptions used.

11. KEY WORDS

Reliability
Testing
Statistical methods and processes